Matrix Exponential using Pauli Matrices

Our aim is to calculate the exponential of Hermitian matrices in $\mathbb{C}^2$ as required for the Stokes matrix. The Pauli matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

form a basis for the Hermitian matrices in $\mathcal{L}(\mathbb{C}^2)$, space of linear operators in $\mathbb{C}^2$. For the above matrices, $\sigma_\alpha^2 = \mathbb{1}$, that is, the σ matrices have eigenvalues $\{\pm 1\}$.

Consider,

$$n \cdot \sigma = \sum_\alpha n_\alpha \sigma_\alpha = n_0 \sigma_0 + n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \tag{2}$$

with $\alpha \in \{0, x, y, z\}$, and,

$$\hat{n} \cdot \vec{\sigma} = \sum_i \hat{n}_i \sigma_i = \hat{n}_x \sigma_x + \hat{n}_y \sigma_y + \hat{n}_z \sigma_z \tag{3}$$

with $i \in \{x, y, z\}$. We define $\hat{n}_i = {}^{n_i}/_{|\vec{n}|}$ and $|\vec{n}| = \sum_i n_i^2$; thus,

$$n \cdot \sigma = n_0 \sigma_0 + |\vec{n}| \hat{n} \cdot \vec{\sigma}. \tag{4}$$

We seek $\mathbf{I} = \exp{(n \cdot \sigma)}$.

For any linear operator $\mathcal{M}$ with complete orthonormal eigenbasis $\{\gamma_i\}$ and eigenvalues $\{\lambda_i\}$, there is a representation $\mathcal{M} = \sum_i \lambda_i \gamma_i \gamma_i^\dagger$. As $\hat{n} \cdot \vec{\sigma}$ has eigenvalues $\{\pm 1\}$ we have

$$\hat{n} \cdot \vec{\sigma} = \gamma_+ \gamma_+^\dagger - \gamma_- \gamma_-^\dagger \tag{5}$$

and thus

$$n \cdot \sigma = (n_0 + |\vec{n}|) \gamma_+ \gamma_+^\dagger + (n_0 - |\vec{n}|) \gamma_- \gamma_-^\dagger. \tag{6}$$

Therefore, we see that,

$$\exp{(n \cdot \sigma)} = e^{n_0 + |\vec{n}|} \gamma_+ \gamma_+^\dagger + e^{n_0 - |\vec{n}|} \gamma_- \gamma_-^\dagger, \tag{7}$$

as for a matrix with orthonormal eigendecomposition $\mathcal{M} = \sum_i \lambda_i \gamma_i \gamma_i^\dagger$ we have for matrix functions: $f(\mathcal{M}) = \sum_i f(\lambda_i) \gamma_i \gamma_i^\dagger$.

We are left with the task of finding the matrices $\gamma_+\gamma_+^\dagger$ and $\gamma_-\gamma_-^\dagger$. Consider the decomposition

$$\hat{n}\cdot\vec{\sigma} = \frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma}) - \frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma}). \tag{8}$$

$\frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma})$ and $\frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma})$ have eigenvalues $\{0,+1\}$ and are projectors onto the subspaces corresponding to $\gamma_+\gamma_+^\dagger$ and $\gamma_-\gamma_-^\dagger$ respectively. We can see this using the identity

$$(\vec{n_1}\cdot\vec{\sigma})(\vec{n_2}\cdot\vec{\sigma}) = (\vec{n_1}\cdot\vec{n_2})\mathbb{1} + (\vec{n_1}\times\vec{n_2})\vec{\sigma} \tag{9}$$

which implies that

$$(\hat{n}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma}) = \mathbb{1} \tag{10}$$

as $\hat{n}\cdot\hat{n}=1$ and $\hat{n}\times\hat{n}=0$. As a consequence, we see the following:

$$\frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma})\frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma}) \tag{11}$$

$$\frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma})\frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma}) \tag{12}$$

$$\frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma})\frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma}) = 0 \tag{13}$$

$$\hat{n}\cdot\sigma\ [{}^1\!/\!{}_2(\mathbb{1}\pm\hat{n}\cdot\vec{\sigma})] = \pm{}^1\!/\!{}_2(\mathbb{1}\pm\hat{n}\cdot\vec{\sigma}) \tag{14}$$

which shows that $\frac{1}{2}(\mathbb{1}\pm\hat{n}\cdot\vec{\sigma})$ are orthogonal projectors onto the subspaces corresponding to eigenvalues $\{\pm1\}$ respectively: ${}^1\!/\!{}_2(\mathbb{1}\pm\hat{n}\cdot\vec{\sigma}) = \gamma_\pm\gamma_\pm^\dagger$.

We therefore see that

$$\begin{aligned}\mathbf{I} &= \exp(n\cdot\sigma)\\ &= e^{n_0+|\vec{n}|}\frac{1}{2}(\mathbb{1}+\hat{n}\cdot\vec{\sigma}) + e^{n_0-|\vec{n}|}\frac{1}{2}(\mathbb{1}-\hat{n}\cdot\vec{\sigma})\\ &= e^{n_0}\left(\cosh|\vec{n}|\mathbb{1}+\sinh|\vec{n}|\hat{n}\cdot\vec{\sigma}\right).\end{aligned} \tag{15}$$

As,

$$n\cdot\sigma = \sum_\alpha n_\alpha\sigma_\alpha = \begin{pmatrix} n_0+n_z & n_x-in_y\\ n_x+in_y & n_0-n_z\end{pmatrix}, \tag{16}$$

we see that

$$\mathbf{I} = e^{n_0}\left(\cosh|\vec{n}|\mathbb{1}+\sinh|\vec{n}|\hat{n}\cdot\vec{\sigma}\right) \tag{17}$$

$$= e^{n_0}\begin{pmatrix}\cosh|\vec{n}|+\sinh|\vec{n}|\hat{n}_z & \sinh|\vec{n}|\hat{n}_x - i\sinh|\vec{n}|\hat{n}_y\\ \sinh|\vec{n}|\hat{n}_x + i\sinh|\vec{n}|\hat{n}_y & \cosh|\vec{n}|-\sinh|\vec{n}|\hat{n}_z\end{pmatrix}. \tag{18}$$