Born rule in Unitary Quantum Mechanics

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Unitary Quantum Mechanics

Axioms of quantum mechanics:

Ax. 1. $|\psi\rangle \in \mathcal{H}, \langle\psi|\psi\rangle = 1$.

Ax. 2. $|\psi\rangle_{\mathcal{SE}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$.

Ax. 3. $|\psi(t)\rangle = \mathcal{U}(t,t')|\psi(t')\rangle$.

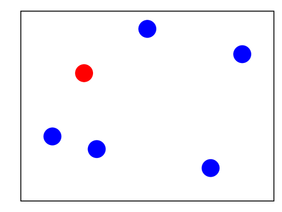
+redundant (like 2nd law of thermodynamics with Newton's laws)

Ax. 4. $\{\Pi_m | \Pi_m = \Pi_m^\dagger, \sum_m \Pi_m = \mathbb{1}, \Pi_m\Pi_n = \delta_{mn}\Pi_m\}$: $\mathcal{P}(m) = \langle\psi\,|\,\Pi_m\,|\,\psi\rangle$ and $|\psi\rangle \xrightarrow{m} \Pi_m|\psi\rangle/\sqrt{\mathcal{P}(m)}$.

Meas. Hyp. 5. $|\Psi\rangle_{\mathcal{SE}} = \sum_m \Psi_m |S_m\rangle_{\mathcal{S}} |E_m\rangle_{\mathcal{E}}$.

MaxEnt Principle

- MaxEnt (maximum entropy) principle: discrete set of choices + no other info $\implies$ uniform probability.
- $\implies$ branches in Meas. Hyp. 5 equiprobable?
- No! discrete set of choices + physics knowledge $\implies$ Born rule.



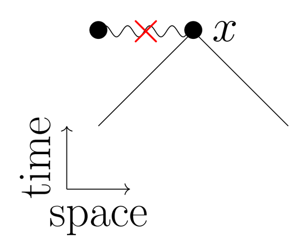
Are red and blue equally likely?

Other Assumptions

Pos. 6. Local (to spacetime point x) properties ${}_x\mathcal{P}^{\mathcal{S}}$ of system $\mathcal{S}$ are function of super-state: ${}_x\mathcal{P}^{\mathcal{S}}(|\Psi\rangle_{\mathcal{SE}})$.

Pos. 7. ${}_x\mathcal{P}^{\mathcal{S}}(|\Psi\rangle_{\mathcal{SE}})$ independent of events outside past light cone of x.

Pos. 8. Physics is relabelling invariant for a system $\mathcal{S}$.



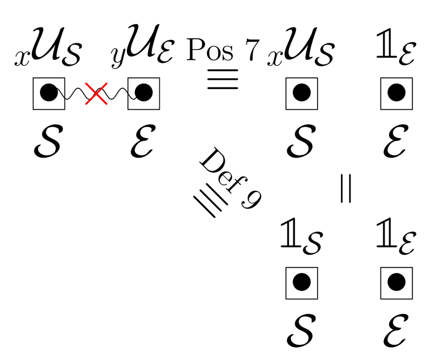
Pos 7 enforces locality.

Zurek's Argument[1]

Def. 9. $\mathcal{U}_{\mathcal{S}}$ envariant wrt $|\Psi\rangle_{\mathcal{SE}} \iff \exists\mathcal{U}_{\mathcal{E}}$: $\mathcal{U}_{\mathcal{S}} \otimes \mathcal{U}_{\mathcal{E}}|\Psi\rangle_{\mathcal{SE}} = |\Psi\rangle_{\mathcal{SE}}$.

Cor. 10. Local (at x) envariant unitary ${}_x\mathcal{U}_{\mathcal{S}}$ does not change system properties: ${}_x\mathcal{P}^{\mathcal{S}}({}_x\mathcal{U}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}}|\Psi\rangle_{\mathcal{SE}}) = {}_x\mathcal{P}^{\mathcal{S}}(|\Psi\rangle_{\mathcal{SE}})$.

Proof.



□

We learn from Zurek [1]:

Lem. 11. With $\sum_m \Psi_m|S_m\rangle_{\mathcal{S}}|E_m\rangle_{\mathcal{E}}$, ${}_x\mathcal{P}^{\mathcal{S}}(|\Psi\rangle) = {}_x\mathcal{P}^{\mathcal{S}}(|\Psi_m||S_m\rangle_{\mathcal{S}}|E_m\rangle_{\mathcal{E}})$ using envariant unitary ${}_x\mathcal{U}^{\mathcal{S}} = \sum_m e^{-\arg\Psi_m}|S_m\rangle\langle S_m|_{\mathcal{S}}$.

Lem. 12. If $|\Psi_m| = |\Psi_n| = \gamma$, then ${}_x\mathcal{P}^{\mathcal{S}}(|\Psi\rangle_{\mathcal{SE}}) = {}_x\mathcal{P}^{\mathcal{S}}(\cdots + \gamma|S_n\rangle_{\mathcal{S}}|E_m\rangle_{\mathcal{E}} \cdots + \gamma|S_m\rangle_{\mathcal{S}}|E_n\rangle_{\mathcal{E}} \ldots)$ using envariant unitary ${}_x\mathcal{U}^{\mathcal{S}} = |S_m\rangle\langle S_n|_{\mathcal{S}} + |S_n\rangle\langle S_m|_{\mathcal{S}} + \sum_{l\neq m,n}|S_l\rangle\langle S_l|_{\mathcal{S}}$.

Thm. 13. $|\Psi_m| = |\Psi_n| \implies \mathcal{P}^{\mathcal{S}}_m(|\Psi\rangle_{\mathcal{SE}}) = \mathcal{P}^{\mathcal{S}}_n(|\Psi\rangle_{\mathcal{SE}})$.

Proof.

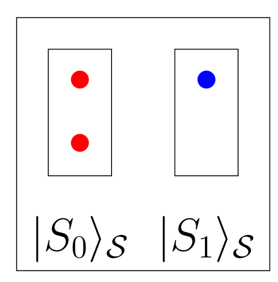
$$\begin{aligned}
&\mathcal{P}^{\mathcal{S}}_i(\ldots \underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{i}\underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{i}\ldots\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{i}\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{j}\ldots) \\
\xrightarrow{Lem\ 12}\ &\mathcal{P}^{\mathcal{S}}_i(\ldots \underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{j}\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{j}\ldots\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{i}\underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{i}\ldots) \\
\xrightarrow{Pos\ 8}\ &\mathcal{P}^{\mathcal{S}}_j(\ldots \underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{i}\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{j}\ldots\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{j}\underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{i}\ldots) \\
\xrightarrow{Lem\ 12}\ &\mathcal{P}^{\mathcal{S}}_j(\ldots \underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{i}\underbrace{|{\color{blue}\blacksquare}\rangle_{\mathcal{S}}}_{i}\ldots\underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{j}\underbrace{|{\color{red}\blacksquare}\rangle_{\mathcal{S}}}_{j}\ldots)
\end{aligned} \tag{1}$$

□

$$|\Psi\rangle_{\mathcal{SE}} = \frac{\sqrt{2}}{\sqrt{3}}|S_0\rangle_{\mathcal{S}}|E_0\rangle_{\mathcal{E}} + \frac{1}{\sqrt{3}}|S_1\rangle_{\mathcal{S}}|E_1\rangle_{\mathcal{E}} \tag{2}$$

and a partition of the environment

$$\begin{aligned}
|E_0\rangle_{\mathcal{E}} &= \frac{1}{\sqrt{2}}|E'_{0,0}\rangle_{\mathcal{E}'}|E''_{0,0}\rangle_{\mathcal{E}''} + \frac{1}{\sqrt{2}}|E'_{0,1}\rangle_{\mathcal{E}'}|E''_{0,1}\rangle_{\mathcal{E}''} \\
|E_1\rangle_{\mathcal{E}} &= |E'_{1,0}\rangle_{\mathcal{E}'}|E''_{1,0}\rangle_{\mathcal{E}''}.
\end{aligned} \tag{3}$$



Physics inspired MaxEnt.

The Schmidt decomposition can now be written as

$$|\Psi\rangle_{\mathcal{SE}'\mathcal{E}''} = \frac{1}{\sqrt{3}}\left[|S_0E'_{0,0}\rangle_{\mathcal{SE}'}|E''_{0,0}\rangle_{\mathcal{E}''} + |S_0E'_{0,1}\rangle_{\mathcal{SE}'}|E''_{0,1}\rangle_{\mathcal{E}''} + |S_1E'_{1,0}\rangle_{\mathcal{SE}'}|E''_{1,0}\rangle_{\mathcal{E}''}\right] \tag{4}$$

reducing to equiprobability case!

Spacetime Density Matrices

In the main text[2] we show the following.

Lem. 14 (Cyclicity of Partial Trace)**.** $\mathrm{tr}_{\mathrm{b}}((\mathbb{1}_{\mathrm{a}} \otimes \mathcal{M}_{\mathrm{b}})\mathcal{T}_{\mathrm{ab}}) = \mathrm{tr}_{\mathrm{b}}(\mathcal{T}_{\mathrm{ab}}(\mathbb{1}_{\mathrm{a}} \otimes \mathcal{M}_{\mathrm{b}}))$.

Lem. 15. $\mathrm{tr}_{\mathrm{b}}((\mathcal{M}_{\mathrm{a}} \otimes \mathbb{1}_{\mathrm{b}})\mathcal{T}_{\mathrm{ab}}(\mathcal{M}'_{\mathrm{a}} \otimes \mathbb{1}_{\mathrm{b}})) = \mathcal{M}_{\mathrm{a}}\,\mathrm{tr}_{\mathrm{b}}(\mathcal{T}_{\mathrm{ab}})\mathcal{M}'_{\mathrm{a}}$.

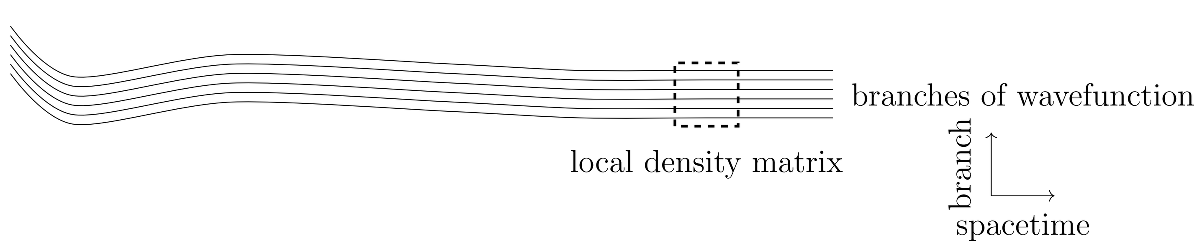
Thm. 16. Consider delocalised space-like separated quantum system a and b. Let the system be in state $|\Psi\rangle_{\mathrm{ab}}$. The reduced density matrix $\rho^{\mathrm{red}}_{\mathrm{a}} = \mathrm{tr}_{\mathrm{b}}(|\Psi\rangle\langle\Psi|_{\mathrm{ab}})$ is only sensitive to the dynamics $\mathcal{U}_{\mathrm{a}}$ at the location of a and not to that at the space-like separated location b.

We further show[2] $\rho^{\mathrm{red}}_{\mathrm{a}} \otimes |\mathcal{A}_0\rangle\langle\mathcal{A}_0|_{\mathrm{c}} \to \mathcal{U}^{\mathrm{meas}}_{\mathrm{ac}}(\rho^{\mathrm{red}}_{\mathrm{a}} \otimes |\mathcal{A}_0\rangle\langle\mathcal{A}_0|_{\mathrm{c}})\mathcal{U}^{\mathrm{meas}\dagger}_{\mathrm{ac}}$ independent of the state and dynamics at b. Tracing out gives:

$$\mathrm{tr}_{\mathrm{c}}\left(\mathcal{U}^{\mathrm{meas}}_{\mathrm{ac}}(\rho^{\mathrm{red}}_{\mathrm{a}} \otimes |\mathcal{A}_0\rangle\langle\mathcal{A}_0|_{\mathrm{c}})\mathcal{U}^{\mathrm{meas}\dagger}_{\mathrm{ac}}\right) = \mathrm{tr}_{\mathrm{c}}\left(\sum_{mm'}\Pi_m\rho^{\mathrm{red}}_{\mathrm{a}}\Pi^\dagger_{m'} \otimes |\mathcal{A}_m\rangle\langle\mathcal{A}_{m'}|_{\mathrm{c}}\right) = \sum_m \Pi_m\rho^{\mathrm{red}}_{\mathrm{a}}\Pi^\dagger_m. \tag{5}$$

for the local dynamics at c.

The spacetime structure along with branching structure can be represented as a figure:



Discussion

- Unitary axioms → unified description of quantum mechanics.
- Unitary axioms + natural postulates → Born rule.
- Equivalent to MaxEnt principle with extra info about quantum universe.
- Rational agent using unitary quantum mechanics uses Born rule.

Thm. 17 (Unitary Measurement Axiom)**.** $\{\Pi_m = \Pi^\dagger_m, \sum_m \Pi_m = \mathbb{1}, \Pi_m\Pi_n = \delta_{mn}\Pi_m\}$: $\mathcal{P}(m) = \mathrm{tr}\,\Pi_m\rho\Pi^\dagger_m$ and $\rho \to \sum_m \Pi_m\rho\Pi^\dagger_m$.

[1] Wojciech Hubert Zurek. "Probabilities from entanglement, Born's rule $p_k = |\,\psi_k\,|^2$ from envariance". In: *Phys. Rev. A* 71 (5 May 2005), p. 052105. DOI: 10.1103/PhysRevA.71.052105. URL: https://link.aps.org/doi/10.1103/PhysRevA.71.052105.

[2] Vishal Johnson et al. *Measurement in a Unitary World*. 2023. arXiv: 2212.03829 [quant-ph].