Compositional Periodicity

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Power and Periodicity

This paper is about the periodicity of functions over composition. It is assumed that the reader is familiar with the concept of composition and with the basic notions of a function. A special notation is adopted for the composition of a function an integral number of times.

Definition. Compositional Power

$$f^{\circ n}(x) = f \circ f \circ \text{..n times..} \circ f(x)$$

Consider the function $f(x) = \frac{1}{x}$. The periodicity of this function is 2. This is because $f(x) \neq x$ but $f \circ f(x) = x$. This leads to a definition of periodicity as below.

Definition. Compositional Periodicity If $f(x) = x$ then the periodicity is 1. Else, let $1 < n$ and, $f^{\circ n} = x$, but for all $0 < m < n$ $f^{\circ m} \neq x$. Then n is the compositional periodicity of the the function f.
In other words, it is the smallest number n such that applying f, n times is equivalent to the identity transform.

Remark. Compositional periodicity shall simply be called periodicity in this paper.

What are the different periodicities that functions can have? First of all, the function $f(x) = x$ has a periodicity of 1. And as seen before, the function $f(x) = \frac{1}{x}$ has a periodicity of 2. A function might not have any periodicity (or equivalently, an infinite periodicity); $f(x) = x + 1$ is an example of such a function.

Consider the function,

$$
\begin{aligned}
f(x) =& \frac{x-1}{x+1}. \\
f \circ f(x) =& \frac{\left(\frac{x-1}{x+1}\right) - 1}{\left(\frac{x-1}{x+1}\right) + 1} = \frac{-1}{x} \\
f \circ f \circ f(x) =& \frac{(\frac{-1}{x}) - 1}{(\frac{-1}{x}) + 1} = \frac{1+x}{1-x}, \text{ and,} \\
f \circ f \circ f \circ f(x) = \frac{(\frac{1+x}{1-x}) - 1}{(\frac{1+x}{1-x}) + 1} = x.
\end{aligned}
$$

Thus, $f(x) = \frac{x-1}{x+1}$ has a periodicity of 4. It shall be described how to get any integral periodicity.

Linear Rational Functions

Consider a linear rational function to be one of the form,

$$A(x) = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}}, a_{21}a_{22} \neq 0.$$

The coefficients can be gathered into a matrix,

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Consider another function,

$$B(x) = \frac{b_{11}x + b_{12}}{b_{21}x + b_{22}}, (b_{21}b_{22} \neq 0)$$

$$\mathcal{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then the composition,

$$
\begin{aligned}
h(x) =& f \circ g(x) = \frac{a_{11}\left(\frac{b_{11}x+b_{12}}{b_{21}x+b_{22}}\right) + a_{12}}{a_{21}\left(\frac{b_{11}x+b_{12}}{b_{21}x+b_{22}}\right) + a_{22}} \\
=& \frac{(a_{11}b_{11} + a_{12}b_{21})x + (a_{11}b_{12} + a_{12}b_{22})}{(a_{21}b_{11} + a_{22}b_{21})x + (a_{21}b_{12} + a_{22}b_{22})}.
\end{aligned}
$$

This just corresponds to the matrix,

$$\mathcal{A} \times \mathcal{B} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Thus, there is an isomorphism between the composition of linear rational functions and multiplication of matrices. This is the key relationship that enables periodicities of any order. Simply consider the rotation matrices!
Consider the rotation matrix,

$$\mathcal{R}_n = \begin{pmatrix} \cos(\frac{\pi}{n}) & -\sin(\frac{\pi}{n}) \\ \sin(\frac{\pi}{n}) & \cos(\frac{\pi}{n}) \end{pmatrix},$$

with the corresponding function,

$$R_n(x) = \frac{\cos(\frac{\pi}{n})x - \sin(\frac{\pi}{n})}{\sin(\frac{\pi}{n})x + \cos(\frac{\pi}{n})}.$$

It is seen that ${\mathcal{R}_n}^n = \mathcal{I}$, whereas for any $0 < m < n$, ${\mathcal{R}_n}^m \neq \mathcal{I}$; where $\mathcal{I}$ is the identity matrix[1]. And, thus, correspondingly, $R^{\circ n}(x) = x$ whereas $R^{\circ m}(x) \neq x$ for any $0 < m < n$. According to the definition given above, this is just the statement that the periodicity of $R(x)$ is n. It is possible to obtain any periodicity whatsoever.

[1] Do note that the function corresponding to $\mathcal{I}$ is just $\frac{1x+0}{0x+1} = x$. Our definitions of identity in the context of matrices and functions are compatible with one another. Of course, this must be true of any isomorphism.