Effective Expansion

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Summary of Prerequisites

Here are summarised some of the results from previous talks that are required in this talk.

The bounds equation [4],

$$\begin{aligned} C^i(x,y) &\leq KM^{(d-2)i}e^{-\delta M^i|x-y|} \\ \partial_{\mu_1}\ldots\partial_{\mu_k}C^i(x,y) &\leq KM^{(d-2+k)i}e^{-\delta M^i|x-y|} \end{aligned} \tag{1}$$

A Brief Overview

This talk continues the discussion of talks 10 through 13 and attempts to arrive at a method of calculation correlation functions.

Talk 13 [6] describes how to isolate a divergent subgraph, of a given graph and scale assignment (G,μ), and replace it with something manageable. Specifically, the amplitude of the graph breaks down into a manageable component and one with a sketchy constant,

$$\begin{aligned} A_{\tilde{G}} &= \int dxdydzdwC(z,y)C(w,y)C(x,y)^2 f(z,w,x) \\ &= (\int dyC(x,y)^2)\int dxdzdwC(z,x)C(z,x)f(z,w,x) \\ &+ \int dxdydzdw(C(z,y)C(w,y)-C(z,x)C(z,x))C(x,y)^2 f(z,w,x) \\ &= \text{sketchy constant} \times \text{manageable integral} + \text{manageable integral} \end{aligned}$$

An extension of this idea to more complicated graphs was hinted at and this is now pursued in the next section.

Formal Localization

Recall that for any graph with a scale assignment, (G,μ), an i-forest of high subgraphs can be generated [3]. Consider the subset of this i-forest containing the divergent high subgraphs and call it $D_\mu(G)$. Define a localization operator τ_g that contracts a graph g to a vertex and replaces all its external propagators with propagators to the reduced vertex $v_e(g, D_\mu)$[1]. In the ϕ^4 theory graphs with $0, 2, 4$ external edges are divergent[2]. The localization operators and reduced vertices can be defined in such a manner as to commute with each other, implying the well-definedness of $\prod_{g \in D_\mu} \tau_g$.

For any maximal subgraph[3] $g \in D_\mu$ choose as $v_e(g, D_\mu)$ any border vertex, one in g connected to an edge external to it. But if there is a possibility to choose a border vertex of the entire graph G, such a choice must be made. Now consider an immediate predecessor $g' \subset g$ in the i-forest:

- If $v_e(g, D_\mu)$ is a border vertex for g' too, choose $v_e(g', D_\mu) = v_e(g, D_\mu)$
- If otherwise g and g' share external edges, choose a vertex connected to one of these as $v_e(g', D_\mu)$
- Otherwise choose any of the remaining border vertices as $v_e(g', D_\mu)$

This is one algorithm to ensure the commutativity of τ_g's. The reason it works is that if $g' \subset g$ share an external edge, it is ensured that τ_g does not move to an internal vertex of g. [4] By now strategically inserting, $1 = \tau_g + (1 - \tau_g)$, the computation of connected correlation functions can be simplified.

The reason for doing all this is seen by looking at the effectively renormalized amplitude, which will be used later,

$$A_G^{ER} = \sum_\mu A_{G,\mu}^{ER} \tag{2}$$

$$A_{G,\mu}^{ER} \equiv \int \prod_\nu dx_\nu R_\mu \prod_l C^{\mu(l)}(x_l, y_l) \tag{3}$$

$$R_\mu \equiv \prod_{g \in D} (1 - \tau_g) \tag{4}$$

The $A_{G,\mu}^{ER}$ are well behaved. Action of τ_g causes for some $x \in g$ and $z \notin g$,

$$C^j(x, y) \to C^j(v_e(g, D), z) + \int_0^1 dt (x - v_e(g, D)) \cdot \nabla C^j(v_e(g, D) + t(x - v_e(g, D)), z)$$

[1] Note that $\tau_g \equiv \tau_g(v_e(g, D_\mu))$ as the vertex that τ_g replaces g with must be specified. In the following sections this indication is supressed for brevity but consistency between these must be noted.

[2] In this and the following section the discussion would be limited to quadruped divergences. Only subgraphs with 4 external edges are addressed here.

[3] The subgraph of lowest scale in any connected subgraph of $D_\mu \subset (G, \mu)i - forest$

[4] Note that the definition of high subgraphs ensures that not only edges, but also no vertices are shared between high subgraphs neither of which is included in the other.

Using the bounds 1, it can be seen that the second term is bounded by $M^{-(i_g(\mu)-e_g(\mu))}$ leading to a similar behaviour and resolution as in the uniform weinberg case[5]. An illuminating example of this procedure was shown in talk 13 [6]. Now comes the main part of this talk which is how to deal with the τ_g.

Effective Expansion in the Biped-Free Case

This talk only deals with the case of quadruped divergence, that is $\omega(g) = 0$. The more complicated cases of $\omega(g) = -2, -4$ are dealt with in [1] and [2].

Firstly, an overall cut-off index ρ, and a bare coupling constant $\mathfrak{g}_\rho$ are defined. It is also useful to define for a vertex v,

$$e_v(\mu) = \max\{\mu(l) | l \text{ hooked to } v\}^5$$

Consider now the bare expansion for a connected schwinger function (N point correlation function),[6]

$$C_N^\rho = \sum_n \frac{(-\mathfrak{g}_\rho)^n}{n!} \sum_{\substack{G, V(G)=n \\ \mu \leq \rho}} A_{G,\mu}, \tag{5}$$

that is, over scale assignments $\mu \in \{0, \ldots \rho\}^{E(G)}$ and over connected graphs at order n with N external edges.

Theorem 1 (Existence of Effective Expansion)**.** *There exist $\rho+1$ formal power series $g_\rho = (g^\rho_{\rho-1}, g^\rho_{\rho-2}, \ldots g^\rho_{-1})$[7] such that the formal power series, equation 5, is the same as:*

$$C_N^\rho = \sum_{G, \mu \leq \rho} \left[\prod_{v \in G} -g^\rho_{e_v(\mu)} \right] A^{ER}_{G,\mu}, \tag{6}$$

where the $A^{ER}_{G,\mu}$ are defined in 3 and the effective constants are recursively defined as,

$$g_i^\rho = g_{i+1}^\rho - \sum_{\substack{quadrupeds\ H \\ \mu \leq \rho, i_H(\mu) = i+1}} \left[\prod_{v \in H} -g^\rho_{e_v(\mu)} \right] \times \int \prod_\nu dx_\nu \left(\prod_{\substack{h \in D_\mu(H) \\ h \neq H}} (1 - \tau_h) \right) (\tau_H) I_{G,\mu} \tag{7}$$

where $i_H(\mu) = i + 1$ indicates that the internal edge of H with smallest scale is $i + 1$ and where,

$$I_{G,\mu} = \prod_{l \in G} C^{\mu(l)}(x, y) \tag{8}$$

[5]Note that this is consistent with defining the vertex v as a graph, $g = \{v\}$.

[6]In the references [1] and [2], the additional indication $()_{,bf}$ is used to indicate that only the biped-free (or rather only the quadruped) case is being discussed. This is suppressed here for brevity.

[7]The upper ρ refers to the global UV cut-off.

is the integrand in the correlation functions.

Proof. [8] The proof proceeds by induction. Consider an intermediate step of the theorem,

$$C_N^\rho = \sum_{G,\mu\leq\rho}\left[\prod_{v\in G} -g^\rho_{\sup(e_v(\mu),i)}\right] A^{ER,i}_{G,\mu} \tag{9}$$

$$A^{ER,i}_{G,\mu} \equiv \int\prod_\nu dx_\nu \prod_{h\in D^i_\mu(G)}(1-\tau_h)I_{G,\mu} \tag{10}$$

$$D^i_\mu(G) \equiv \{h\in D_\mu(G)|i_h(\mu)>i\} \tag{11}$$

The claim is that C_N^ρ defined at each stage over i equals that at any other stage. First to note are the boundary cases. As there is an overall UV cutoff of ρ, equation 9 reduces to equation 5 for $i=\rho$[9]. In case of $i=-1$, it reduces to equation 6.

Now all that needs to be shown is the inductive step. In going from $i+1$ to i the change in $A^{ER,\cdot}_{G,\mu}$ are those involving,

$$\prod_{h\in D^i_\mu\backslash D^{i+1}_\mu}(1-\tau_h),$$

the subgraphs that need to be reduced while passing from stage $i+1$ to stage i. These are the terms corresponding to $\{H_1,H_2,\dots H_k\}=\{H\in D_\mu|i_H=i+1\}$.

This can be achieved by adding to and subtracting from $A^{ER,i+1}_{G,\mu}$ the quantities,

$$\sum_{\substack{S\subset\{H_1,H_2\dots H_k\}\\ S\neq\phi}}\prod_{H\in S}(-\tau_H)\prod_{h\in D^{i+1}_\mu}(1-\tau_h)I_{G,\mu} \tag{12}$$

The added term now includes everything required for $A^{ER,i}_{G,\mu}$[10]. What about the subtracted term? Some more terminology is needed to deal with this unfor-

[8] The proof given here is very skin and bones, just to highlight the essential points. For a more detailed look please check out [1] and [2].

[9] Here it must be specified that g^ρ_ρ is taken to be g_ρ, the bare propagator.

[10] The terms in $\prod_{h\in D^i_\mu}$ but not in $\prod_{h\in D^{i+1}_\mu}$ are precisely $\prod_{h\in\{H_1,\dots H_k\}}(1-\tau_h)$. The term corresponding to $S=\phi$ are already included in $\prod_{h\in D^i_\mu}$ and all the other terms come from the added term here.

tunately. For $S \subset D_\mu^i \setminus D_\mu^{i+1}$,

$$A^{ER,i}_{G,\mu,S} \equiv A^{ER,i}_{G,\mu} \qquad\qquad \Longleftarrow S = \phi \tag{13}$$

$$A^{ER,i}_{G,\mu,S} \equiv \int \prod_\nu d\nu \prod_{H\in S}(-\tau_H) \prod_{h\in D_\mu^{i+1}} (1-\tau_h) I_{G,\mu} \qquad \Longleftarrow \text{otherwise} \tag{14}$$

$$\Longrightarrow C_N^\rho = \sum_{\substack{(G,\mu,S)\\ \mu\le\rho, S\subset D_\mu^i\setminus D_\mu^{i+1}}} \left[\prod_{v\in G} -g^\rho_{\sup(e_v(\mu),i+1)}\right] A^{ER,i}_{G,\mu,S}, \tag{15}$$

as S now contains all subsets with $S = \phi$ corresponding to the $A^{ER,i}_{G,\mu}$ term and the "subracted" terms all coming from the other sum.[11]

Define another (sigh) operator $\varphi_i(G,\mu,S) = (G',\mu',\phi)$ that replaces each divergent subgraph, $H \in S$, with its corresponding single vertex $v_e(H,\mu)$. It also updates the scales μ to μ' consistently so that the edges remaining in G' inherit the correct weights from G. One more thing to be noted is that for all of these subgraphs as $i_H(\mu) = i+1$, $e_{v_e}(\mu') = e_{v_e}(\mu) \le i$, the replaced vertices are all of scale $\ge i+1$. Now rewrite equation 15 as,

$$C_N^\rho = \sum_{(G',\mu')} \left\{ \sum_{\substack{(G,\mu,S),\mu\le\rho\\ \varphi_i(G,\mu,S)=(G',\mu',\phi)}} \left[\prod_{v\in G} -g^\rho_{\sup(e_v(\mu),i+1)}\right] A^{ER,i}_{G,\mu,S} \right\} \tag{16}$$

and compare with equation 7 to complete the induction step. □

Remarks. Some remarks regarding the expansions.

- There are two entities being renormalized, the couplings g on the one hand and the amplitudes A^{ER} on the other.
- The renormalization only works at the level of contractions. The overall sum over the graphs (G,μ) and not individual summands are what are being renormalized.
- Inspite of this, the sums are well defined as there is an overall UV cutoff ρ that is specified.

[11] A very important point to notice here is that it is not the amplitude $A^{ER,\cdot}_{G,\mu}$ that is being update but the whole C_N^ρ. So, this argument only works at the level of contraction schemes.

Statement of General Theorem and Remarks

Theorem 2 (Gen. Theorem)**.** *There exist* $3(\rho+1)$ *formal power series in* g_ρ *called* g_i*,* δM_i^2 *and* $\delta\mathcal{Z}_i$ *with* $i \in \{\rho-1, \rho-2\cdots-1\}$ *such that the formal power series for* C_N^ρ *can be written as,*

$$C_N^\rho = \sum_{\hat{G},\mu\le\rho}\left[\prod_{v\in V(\hat{G})} -g_{e_v(\mu)}\right]\left[\prod_{w\in W^0(\hat{G})} -\delta M^2_{e_2 w(\mu)}\right]\left[\prod_{w\in W^1(\hat{G})} -\delta\mathcal{Z}_{e_w(\mu)}\right] A^{ER}_{\hat{G},\mu}, \tag{17}$$

with,

$$A^{ER,i}_{G,\mu} \equiv \int \prod_{\nu\in V\cup W^0\cup W^1} dx_\nu \left(\prod_{h\in D^i_\mu}(1-\tau_h)\prod_{w\in W^1(\hat{G})} -\Delta\right) I_{\hat{G},\mu} \tag{18}$$

Remarks. Some remarks about the general theorem.

- The multiscale renormalization procedure handles the divergence issues of the integrals but fails to control the growth of graphs. The number of possible subgraphs is still infinite.

- To calculate the coupling at level i, not only is the coupling at level $i+1$ required but all higher couplings as well. There is an indication in the reference [1] that this "non-Markovianness" can be overcome using renormalons.

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References

[1] Razvan Gurau, Vincent Rivasseau and Alessandro Sfondrini, *Renormalization: an advanced overview*, 2014.

[2] Vincent Rivasseau, *From perturbative to constructive renormalization*, 1991.

[3] Richard Milbradt, *Scales and slicing. High subgraphs and i-forests*, 2021.

[4] Janik Kruse, *Superficial Degree of Divergence*, 2021.

[5] Sebastian Albrecht, *Uniform Weinberg Theorem*, 2021.

[6] Damiano Aliverti-Piuri, *Localization*, 2021.