Generalised Measurement

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This article tries to explore the formalism of generalised measurements in quantum mechanics and whether they can be represented as projective measures in a larger hilbert space. This article is strongly influenced by a lecture [1].

Motivation

Many axiomatisations of quantum mechanics have a measurement postulate[2]. It specifies the probability of measuring the system to be in a certain state and the post-measurement state. Consider the system to start out in the state,

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where the hilbert space is,

$$\mathcal{H}_2 = span\{|0\rangle, |1\rangle\}, \tag{1}$$

which is just a qubit state.

The usual bornesque axiom is to assume that the probability to obtain state $|0\rangle$ is $|\alpha|^2$ and that the post-measurement state is simply $|0\rangle$.

$$\mathcal{P}(0) = |\alpha|^2 = \langle 0|\psi\rangle \langle\psi|0\rangle = \langle 0| \rho |0\rangle = \mathrm{tr}(|0\rangle \langle 0| \rho |0\rangle \langle 0|)$$

where $\mathcal{P}$ is the probability of obtaining state 0 and ρ is the density matrix corresponding to $|\psi\rangle$[1] Also, the post measurement state is given by,

$$\rho_0' = |0\rangle \langle 0| = \frac{|0\rangle \langle 0| \rho |0\rangle \langle 0|}{\mathrm{tr}(|0\rangle \langle 0| \rho |0\rangle \langle 0|)}.$$

[1] Here $\rho = |\psi\rangle \langle\psi|$. Everything done here works directly for density matrices generally so they would be used in the remainder of the article.

This means that for a generic hilbert space $\mathcal{H} = span\{|i\rangle\}$, defining the projector operators as[2],

$$\mathbb{P}_m = |m\rangle \langle m| ,$$

the probability of and state after measurement outcome m is,

$$\mathcal{P}(m) = \mathrm{tr}\big(\mathbb{P}_m \rho \mathbb{P}_m^\dagger\big)$$
$$\rho'_m = \frac{\mathbb{P}_m \rho \mathbb{P}_m^\dagger}{\mathrm{tr}\Big(\mathbb{P}_m \rho \mathbb{P}_m^\dagger\Big)} = \frac{\mathbb{P}_m \rho \mathbb{P}_m^\dagger}{\mathcal{P}(m)},$$

and this is the usual projective measurement.

Generalised Measurements and POVMs[3]

A set of generalised measure operators is $S_{\mathbb{M}} = \{\mathbb{M}_m\}$ where the operators need no longer be hermitian or orthogonal but need to satisfy,

$$\mathbb{1} = \sum_m \mathbb{M}_m^\dagger \mathbb{M}_m. \tag{2}$$

The probability and post-measurement state are now given by,

$$\mathcal{P}_{S_{\mathbb{M}}}(m) = \mathrm{tr}\big(\mathbb{M}_m \rho \mathbb{M}_m^\dagger\big)$$
$$\rho'_m = \frac{\mathbb{M}_m \rho \mathbb{M}_m^\dagger}{\mathcal{P}(m)},$$

and the normalisation of probability is maintained (using equation 2),

$$\sum_m \mathcal{P}_{S_{\mathbb{M}}}(m) = \sum_m \mathrm{tr}\big(\mathbb{M}_m \rho \mathbb{M}_m^\dagger\big) = \mathrm{tr}\Bigg((\sum_m \mathbb{M}_m^\dagger \mathbb{M}_m)\rho \Bigg) = \mathrm{tr}(\rho) = 1.$$

For the given set of measure operators one can define the set of povms as,

$$S_{\mathbb{E}} = \{\mathbb{E}_m\} = \{\mathbb{M}_m^\dagger \mathbb{M}_m\},$$

which therefore satisfy,

$$\mathbb{1} = \sum_m \mathbb{E}_m. \tag{3}$$

But also note that the operators are now positive (which need not be the case for the measure operators $S_{\mathbb{M}}$). That is,

$$\forall\, |\psi\rangle \in \mathcal{H}, \langle\psi| \,\mathbb{E}_m\, |\psi\rangle \geq 0.$$

[2]Note that these are orthogonal projection operators. They are hermitian.
[3]Standing for Positive Operator Valued Measures

Given any set of positive operators $S_{\mathbb{E}}$ that satisfy equation 3, they form a set of povms with,

$$\mathcal{P}_{S_{\mathbb{E}}}(m) = \operatorname{tr}(\mathbb{E}_m \rho).$$

Also, a set of generalised measure operators can always be found for a given set of povms. As the operators in $S_{\mathbb{E}}$ are positive, they are hermitian and diagonalizable meaning they can be expanded in a diagonal orthonormal basis with positive coefficients α_i as,

$$\mathbb{E}_m = \sum_i \alpha_i \left|e_i\right\rangle \left\langle e_i\right|$$

which means that a set $S_{\mathbb{M}}$ defined as,

$$\mathbb{M}_m = \sum_i \sqrt{\alpha_i} \left|e_i\right\rangle \left\langle e_i\right|$$

gives rise to the given set of povms $S_{\mathbb{E}}$.

Now to check if they give the same statistics,

$$\mathcal{P}_{S_{\mathbb{M}}}(m) = \operatorname{tr}\left(\mathbb{M}_m \rho \mathbb{M}_m^\dagger\right) = \operatorname{tr}\left(\mathbb{M}_m^\dagger \mathbb{M}_m \rho\right) = \operatorname{tr}(\mathbb{E}_m \rho) = \mathcal{P}_{S_{\mathbb{E}}}(m).$$

Which they do seem to do

This means that the description of a certain physical phenomenon with measure operators is equivalent to one with povms in case all that is used are the statistics of experiment results.

Why they are lacking

The povms and measure operators are not in a one-one correspondence. Consider the set of measure operators $S_{\tilde{\mathbb{M}}} = \{U\mathbb{M}_m\}$ for some set $S_{\mathbb{M}}$ and some unitary U[4]. Then,

$$S_{\tilde{\mathbb{E}}} = \{\tilde{\mathbb{M}}^\dagger \tilde{\mathbb{M}}\} = \{\mathbb{M}^\dagger \mathbb{M}\} = S_{\mathbb{E}},$$

and so both sets of measure operators give rise to the same set of povms.

As they do give rise to the same statistics however, is this unitary freedom of any consequence? While the statistics for a single experiment are the same, the statistics for multiple experiments would differ. This shall be explained using the example of $\mathcal{H}_2$ defined above (equation 1).

[4] This is well defined as the unitary and measure operators belong to the space of linear operators on $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$.

Consider the sets of measure operators $S_{\mathbb{M}}$ and $S_{\tilde{\mathbb{M}}}$

$$\mathbb{M}_0 = |0\rangle\langle 0| \qquad \mathbb{M}_1 = |1\rangle\langle 1|$$
$$\tilde{\mathbb{M}}_0 = |1\rangle\langle 0| \qquad \tilde{\mathbb{M}}_1 = |0\rangle\langle 1|.$$

It can be directly seen that they are unitarily related and the povms obtained from them are the same.

Consider the density matrix, $\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$. The two measure operators of course give rise to the same statistics as they give rise to the same povms,

$$\mathcal{P}_{S_{\mathbb{M}}}(0) = \frac{1}{3} = \mathcal{P}_{S_{\tilde{\mathbb{M}}}}(0).$$

However, the post-measurement state is different in the two cases,

$$\rho_0' = \frac{\mathbb{M}_0\rho\mathbb{M}_0^\dagger}{\mathcal{P}_{S_{\mathbb{M}}}(0)} = |0\rangle\langle 0|$$
$$\tilde{\rho}_0' = \frac{\tilde{\mathbb{M}}_0\rho\tilde{\mathbb{M}}_0^\dagger}{\mathcal{P}_{S_{\tilde{\mathbb{M}}}}(0)} = |1\rangle\langle 1|$$

which gives different statistics for the second experiment.

In fact the probabilities of obtaining 0 twice are not equal,

$$\mathcal{P}_{S_{\mathbb{M}}}(0,0) = \frac{1}{3} \qquad \neq \qquad 0 = \mathcal{P}_{S_{\tilde{\mathbb{M}}}}(0,0)$$

which seems to indicate that while povms are well defined for a single experiment, they are not well defined while conducting multiple experiments. In order for the post-measurement state to be well defined, generalised measure operators must be used.

POVMs as Projections

This section shows that a povm on a certain hilbert space gives the same one-experiment statistics as projective measurments in another (sometimes larger) hilbert space. An article by Peter Shor[4] was used as reference.

For the rest of the articles a hilbert space of d-dimensions, $\mathcal{H} = span\{|i\rangle\}_{i\in\{0,1\ldots d-1\}}$, shall be fixed (unless otherwise specified). Not that this is the hilbert space of the actual system, the projective measure hilbert space is sometimes larger.

Rank-1

First consider the case where all the povms are rank-1 operators. So generically consider the case $S_{\mathbb{E}}$ with,

$$\mathbb{E}_k = \lambda_k \left|\psi_k\right\rangle \left\langle\psi_k\right| .$$

Here the $|\psi_k\rangle$ are assumed to be normalised so that $\langle\psi_k|\psi_k\rangle = 1$[5]. The λ_ks need to be positive so that positive operators obtain. However, the number of povms, $N = |S_{\mathbb{E}}| = \sum_m \operatorname{rank} \mathbb{E}_m$, need not be the same as d, in fact it is generally different from d[6].

Redefine the states so that,

$$\mathbb{E}_k = \lambda_k \left|\psi_k\right\rangle \left\langle\psi_k\right| = \left|\psi'_k\right\rangle \left\langle\psi'_k\right| ,$$

with $|\psi'_k\rangle = \sqrt{\lambda_k}\,|\psi_k\rangle$.

Normalisation condition for the set $S_{\mathbb{E}}$ can be rephrased as,

$$\mathbb{1} = \sum_k \left|\psi'_k\right\rangle \left\langle\psi'_k\right| .$$

Now consider the canonical orthonormal basis of $\mathcal{H}$, $\{|i\rangle\}_{i\in\{0,1\ldots d-1\}}$. Look at the matrix,

$$\underbrace{\mathcal{F}}_{(d\times N)} = \begin{pmatrix} \langle 0|\psi'_0\rangle & \langle 0|\psi'_1\rangle & \cdots & \left\langle 0\middle|\psi'_{N-1}\right\rangle \\ \langle 1|\psi'_0\rangle & \langle 1|\psi'_1\rangle & \cdots & \left\langle 1\middle|\psi'_{N-1}\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle d-1|\psi'_0\rangle & \langle d-1|\psi'_1\rangle & \cdots & \left\langle d-1\middle|\psi'_{N-1}\right\rangle \end{pmatrix},$$

with $\mathcal{F}_{ij} = \left\langle i\middle|s\psi'_j\right\rangle$.

Note,

$$\delta_{kl} = \langle k|l\rangle = \langle k|\sum_\alpha \mathbb{E}_\alpha \left|l\right\rangle = \sum_\alpha \langle k|\psi'_\alpha\rangle \langle\psi'_\alpha|l\rangle$$

which means that the rows of $\mathcal{F}$ are orthonormal and can thus be extended to a unitary $\underbrace{\tilde{\mathcal{F}}}_{N\times N}$ using the Gram-Schmidt procedure[3], say,

$$\tilde{\mathcal{F}} = \begin{pmatrix} & & \mathcal{F} & \\ \tilde{\mathcal{F}}_{d0} & \tilde{\mathcal{F}}_{d1} & \cdots & \tilde{\mathcal{F}}_{dN-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathcal{F}}_{N-10} & \tilde{\mathcal{F}}_{N-11} & \cdots & \tilde{\mathcal{F}}_{N-1N-1} \end{pmatrix}$$

[5]However, they need not be an orthonormal set. $\langle\psi_j|\psi_k\rangle$ need not be δ_{jk}.

[6]However, $N \overset{!}{\geq}$ d as $\mathbb{1}$ spans the entire hilbert space and so must $S_{\mathbb{E}}$.

so that,

$$\tilde{\mathcal{F}}\tilde{\mathcal{F}}^\dagger = \mathbb{1} \implies \tilde{\mathcal{F}}^\dagger\tilde{\mathcal{F}} = \mathbb{1},$$

which means that the columns of $\tilde{\mathcal{F}}$ are also orthonormal.

Extend $\mathcal{H}$ to $\tilde{\mathcal{H}} = span\{|i\rangle\}_{i\in\{0,1\ldots N-1\}} = \mathcal{H} \oplus span\{|i\rangle\}_{i\in\{d\ldots N-1\}}$ and define,

$$\begin{aligned}\underbrace{\left|\tilde{\psi}_i\right\rangle}_{\in\tilde{H}} &= \sum_j \tilde{\mathcal{F}}_{ji}\,|j\rangle = \sum_{j=0}^{d-1} |j\rangle\,\langle j|\psi_i'\rangle + \sum_{j=d}^{N-1} |j\rangle\,\tilde{\mathcal{F}}_{ji} \\ &= \underbrace{|\psi_i'\rangle}_{\in\mathcal{H}} + \sum_{j=d}^{N-1} |j\rangle\,\tilde{\mathcal{F}}_{ji}\end{aligned}$$

Note that $\left|\tilde{\psi}_i\right\rangle$ are orthonormal as $\tilde{\mathcal{F}}$ is unitary. Consider measure operators,

$$S_{\tilde{\mathbb{P}}} = \{\tilde{\mathbb{P}}_j\} = \{\left|\tilde{\psi}_j\right\rangle\left\langle\tilde{\psi}_j\right|\},$$

which are orthogonal projectors in $\tilde{\mathcal{H}}$[7]

Consider a state $\rho \in \mathcal{H} \subset \tilde{\mathcal{H}}$ with,

$$\rho \in span\{|i\rangle\,\langle j|\}_{i,j\in\{0\ldots d-1\}}$$

$$\begin{aligned}\implies \operatorname{tr}\left(\mathbb{P}_j\rho\mathbb{P}_j^\dagger\right) = \operatorname{tr}\left(\left|\tilde{\psi}_j\right\rangle\left\langle\tilde{\psi}_j\right|\rho\left|\tilde{\psi}_j\right\rangle\left\langle\tilde{\psi}_j\right|\right) &= \left\langle\tilde{\psi}_j\right|\rho\left|\tilde{\psi}_j\right\rangle \\ &= \underbrace{\langle\psi_j'|\,\rho\,|\psi_j'\rangle}_{\text{as } \rho\in\mathcal{H}} \\ &= \operatorname{tr}\left(|\psi_j'\rangle\,\langle\psi_j'|\,\rho\right) \\ &= \operatorname{tr}(\mathbb{E}_j\rho)\end{aligned}$$

$$\implies \mathcal{P}_{S_{\tilde{\mathbb{P}}}}(j) = \mathcal{P}_{S_{\mathbb{E}}}(j)$$

which means projective measurements in $\tilde{\mathcal{H}}$ give the same statistics as povms in $\mathcal{H}$!

General Rank

Now consider povm operators of general rank that obey positivity and normalisation,

$$\sum_k \mathbb{E}_k = \mathbb{1} \text{ and } \mathbb{E}_k \geq 0.$$

[7] as $\tilde{\mathbb{P}}_i\tilde{\mathbb{P}}_j = \delta_{ij}\tilde{\mathbb{P}}_j$.

as $\mathbb{E}_k \geq 0$, $\exists$ an orthonormal diagonalisation,

$$\mathbb{E}_k = \sum_{l_k} \lambda_{l_k} |e_{l_k}\rangle \langle e_{l_k}| = \sum_{l_k} \left|e'_{l_k}\right\rangle \left\langle e'_{l_k}\right|$$

with $\left|e'_{l_k}\right\rangle = \sqrt{\lambda_{l_k}}\, |e_{l_k}\rangle$.

$$\implies \sum_k \sum_{l_k} \left|e'_{l_k}\right\rangle \left\langle e'_{l_k}\right| = \mathbb{1}$$

Let $N = \sum_k \operatorname{rank} \mathbb{E}_k$ and consider[8],

$$\underbrace{\mathcal{F}}_{(d\times N)} = \begin{pmatrix} \langle 0|e'_0\rangle & \langle 0|e'_1\rangle & \dots & \left\langle 0\middle|e'_{N-1}\right\rangle \\ \langle 1|e'_0\rangle & \langle 1|e'_1\rangle & \dots & \left\langle 1\middle|e'_{N-1}\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle d-1|e'_0\rangle & \langle d-1|e'_1\rangle & \dots & \left\langle d-1\middle|e'_{N-1}\right\rangle \end{pmatrix},$$

with $\mathcal{F}_{il_k} = \mathcal{F}_{i\alpha} = \langle i|e'_\alpha\rangle$.

Now extend this to $\tilde{\mathcal{H}} = span\{|i\rangle\}_{i\in\{0...N-1\}} \supset \mathcal{H}$. And let, for unitary $\tilde{\mathcal{F}}$,

$$\tilde{\mathcal{F}} = \begin{pmatrix} & & \mathcal{F} & \\ \tilde{\mathcal{F}}_{d0} & \tilde{\mathcal{F}}_{d1} & \dots & \tilde{\mathcal{F}}_{dN-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathcal{F}}_{N-10} & \tilde{\mathcal{F}}_{N-11} & \dots & \tilde{\mathcal{F}}_{N-1N-1} \end{pmatrix}$$

and,

$$\begin{aligned} \underbrace{|\tilde{e}_{k_i}\rangle}_{\in\tilde{H}} = \sum_j \tilde{\mathcal{F}}_{jk_i} |j\rangle &= \sum_{j=0}^{d-1} |j\rangle \left\langle j\middle|e'_{k_i}\right\rangle + \sum_{j=d}^{N-1} |j\rangle\, \tilde{\mathcal{F}}_{jk_i} \\ &= \underbrace{\left|e'_{k_i}\right\rangle}_{\in\mathcal{H}} + \sum_{j=d}^{N-1} |j\rangle\, \tilde{\mathcal{F}}_{jk_i}. \end{aligned}$$

Note that $\{|\tilde{e}_{k_i}\rangle\}$ are orthonormal which means that $|\tilde{e}_{k_i}\rangle \langle\tilde{e}_{k_i}|$ are projectors and so are,

$$\tilde{\sqcap}_i = \sum_{k_i} |\tilde{e}_{k_i}\rangle \langle \tilde{e}_{k_i}|,$$

leading to the set of orthonormal projectors $S_{\tilde{\sqcap}} = \{\tilde{\sqcap}_i\}$.

[8] For simplicity of notation the double index notation l_k would sometimes be represented with a simple greek letter α with an implicit bijection between the sets $\{l_k\}$ and $\{0...N-1\}$. It should be clear from context what is being referenced.

Let $\rho \in \mathcal{H} \subset \tilde{\mathcal{H}}$,

$$\begin{aligned}\implies \mathcal{P}_{S_{\tilde{\Pi}}}(i) = \operatorname{tr}\left(\tilde{\Pi}_i \rho\right) = \sum_{k_i} \operatorname{tr}(|\tilde{e}_{k_i}\rangle \langle \tilde{e}_{k_i}| \rho) &= \sum_{k_i} \langle \tilde{e}_{k_i}| \rho |\tilde{e}_{k_i}\rangle \\ &= \underbrace{\sum_{k_i} \langle e'_{k_i}| \rho |e'_{k_i}\rangle}_{\text{as } \rho \in \mathcal{H}} \\ &= \operatorname{tr}\left((\sum_{k_i} |e'_{k_i}\rangle \langle e'_{k_i}|) \rho \right) \\ &= \operatorname{tr}(\mathbb{E}_k \rho) = \mathcal{P}_{S_{\mathbb{E}}}(i)\end{aligned}$$

For a single experiment, same statistics obtain! Povms on a certain hilbert space give the same statistics as projective measurments on a larger space.

Generalised Measurements as Projections

This section explores whether even the post-measurement state can be obtained through projective measurments.

Consider generalised measure operators $S_{\mathbb{M}} = \{\mathbb{M}_m\}$. Consider a singular value decomposition of these measure operators [3],

$$\mathbb{M}_m = \sum_{i_m} \sigma_{i_m} |u_{i_m}\rangle \langle v_{i_m}|, \sigma_{i_m} \geq 0 \in \mathbb{R}$$

$$\begin{aligned}\mathcal{P}_{S_{\mathbb{M}}}(m) = \operatorname{tr}\left(\mathbb{M}_m \rho \mathbb{M}_m^\dagger\right) &= \operatorname{tr}\left(\sum_{i_m} \sum_{j_m} \sigma_{i_m} \sigma_{j_m} |u_{i_m}\rangle \langle v_{i_m}| \rho |v_{j_m}\rangle \langle u_{j_m}| \right) \\ &= \sum_{i_m} \sigma_{i_m}^2 \langle v_{i_m}| \rho |v_{i_m}\rangle .\end{aligned}$$

Consider the operators $|v'_{i_m}\rangle = \sigma_{i_m} |v_{i_m}\rangle$, which implies,

$$\mathcal{P}_{S_{\mathbb{M}}}(m) = \sum_{i_m} \langle v'_{i_m}| \rho |v'_{i_m}\rangle . \tag{4}$$

and

$$\begin{aligned}\mathbb{1} = \sum_m \mathbb{M}_m^\dagger \mathbb{M}_m &= \sum_m \sum_{i_m} \sum_{j_m} \left|v'_{i_m}\right\rangle \langle u_{i_m}|u_{j_m}\rangle \left\langle v'_{j_m}\right| \\ &= \sum_m \sum_{i_m} \left|v'_{i_m}\right\rangle \left\langle v'_{i_m}\right| \\ \implies \delta_{kl} = \langle k|\,\mathbb{1}\,|l\rangle &= \sum_m \sum_{i_m} \left\langle k\middle|v'_{i_m}\right\rangle \left\langle v'_{i_m}\middle|l\right\rangle\end{aligned}$$

Hence a matrix $\mathcal{F}_{ij_m} = \left\langle i\middle|v'_{j_m}\right\rangle$ with orthonormal rows can be defined and can be extended to unitary,

$$\tilde{\mathcal{F}} = \begin{pmatrix} & & \mathcal{F} & \\ \tilde{\mathcal{F}}_{d0} & \tilde{\mathcal{F}}_{d1} & \dots & \tilde{\mathcal{F}}_{dN-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathcal{F}}_{N-10} & \tilde{\mathcal{F}}_{N-11} & \dots & \tilde{\mathcal{F}}_{N-1N-1} \end{pmatrix}$$

where $N = \sum_m \operatorname{rank} \mathbb{M}_m$ and a hilbert space $\tilde{\mathcal{H}} = span\{|i\rangle\}_{i\in\{0\dots N-1\}}$ can be defined.

Now define,

$$\underbrace{|\tilde{v}_{i_m}\rangle}_{\in\tilde{\mathcal{H}}} = \sum_j \tilde{\mathcal{F}}_{ji_m}\,|j\rangle = \underbrace{\sum_{j=0}^{d-1} |j\rangle \left\langle j\middle|v'_{i_m}\right\rangle}_{\in\mathcal{H}} + \sum_{j=d}^{N-1} \tilde{\mathcal{F}}_{ji_m}\,|j\rangle$$

and,

$$\tilde{\Pi}_m = \sum_{i_m} |\tilde{v}_{i_m}\rangle \langle\tilde{v}_{i_m}|\,.$$

For $\rho \in \mathcal{H}$,

$$\begin{aligned}\mathcal{P}_{S_{\tilde{\Pi}}}(m) = \operatorname{tr}\left(\tilde{\Pi}_m\rho\right) &= \sum_{i_m} \langle\tilde{v}_{i_m}|\,\rho\,|\tilde{v}_{i_m}\rangle \\ &= \sum_{i_m} \left\langle v'_{i_m}\right|\rho\left|v'_{i_m}\right\rangle \underset{\text{equation 4}}{=} \mathcal{P}_{S_{\mathbb{M}}}(m)\end{aligned}$$

The post measurement state

In the larger hilbert space $\tilde{\mathcal{H}}$, the post-measurement state is,

$$\begin{aligned}\propto \tilde{\Pi}_m\rho\tilde{\Pi}_m^\dagger &= \sum_{i_m}\sum_{j_m} |\tilde{v}_{i_m}\rangle \langle\tilde{v}_{i_m}|\,\rho\,|\tilde{v}_{j_m}\rangle \langle\tilde{v}_{j_m}| \\ &= \sum_{i_m}\sum_{j_m} |\tilde{v}_{i_m}\rangle \left\langle v'_{i_m}\right|\rho\left|v'_{j_m}\right\rangle \langle\tilde{v}_{j_m}|\,.\end{aligned}$$

Consider operator,

$$\sum_{i_m} |u_{i_m}\rangle \langle\tilde{v}_{i_m}| = \mathcal{U}_m : \tilde{\mathcal{H}} \to \mathcal{H},$$

which acting on the post-measurement state above gives, noting that $\delta_{i_m j_m} = \langle\tilde{v}_{i_m}|\tilde{v}_{j_m}\rangle$,

$$\begin{aligned}\mathcal{U}_m\tilde{\mathbb{\Pi}}_m\rho\tilde{\mathbb{\Pi}}_m^\dagger\mathcal{U}_m^\dagger &= \sum_{i_m}\sum_{j_m} |u_{i_m}\rangle \langle v'_{i_m}| \rho |v'_{j_m}\rangle \langle u_{j_m}| \\ &= (\sum_{i_m} |u_{i_m}\rangle \langle v'_{i_m}|)\rho(\sum_{j_m} |v'_{j_m}\rangle \langle u_{j_m}|) \\ &= \mathbb{M}_m\rho\mathbb{M}_m^\dagger.\end{aligned}$$

There are also operators that go "back",

$$\sum_{i_m} |\tilde{v}_{i_m}\rangle \langle u_{i_m}| = \mathcal{U}_m^{\text{"}-1\text{"}} : \mathcal{H} \to \tilde{\mathcal{H}},$$

so that,

$$\mathcal{U}_m^{\text{"}-1\text{"}}\mathbb{M}_m\rho\mathbb{M}_m^\dagger\mathcal{U}_m^{\text{"}-1\text{"}\dagger} = \tilde{\mathbb{\Pi}}_m\rho\tilde{\mathbb{\Pi}}_m^\dagger,$$

and one can freely move between $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

This means that projective measurements in $\tilde{\mathcal{H}}$ are equivalent to generalised measure operators in $\mathcal{H}$.

References

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