Slider

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The idea of sliders was inspired by situations in which the fuzziness in human thinking plays a role. For example, rating students on a subjective examination is not a clear-cut crisp system of evaluation but rather depends a lot on the evaluator's biases and preconceptions. If the process is made too mechanical and a very strict rubric is used, it would lose the subjective characteristic that sets it apart from multiple choice exams. If on the other hand a very loose rubric is used, there is the problem of the system becoming unfair. Slider grew as an attempt to address this problem.

The Real Numbers

The idea behind sliders is to use a continuous set of numbers in order to instill the property of fuzziness. A discrete set of numbers always involves a jump between any two consecutive values and this non-continuity leads to an inability to truly incorporate fuzziness. It is thus required that a continuous evaluation metric be used and this is the issue that this paper attempts to solve.

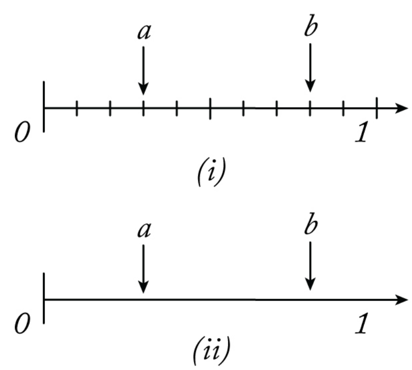

Figure 1: (i) A discrete measure can only take on a discrete set of value, and thus there can be no fuzziness. (ii) Continuous measures can be fuzzy.

The real numbers are as good a solution as any other in this regard, the only problem is when one makes a decimal approximation of a real number; this leads to it becoming a discrete metric once again, thereby losing its fuzziness. One solution to this problem is to let the 'real number measure' correspond to a geometrical entity such as the length of a line segment, and therefore let the line segment represent the real number. The only way to measure the length of the line segment, and determine the real number it encodes, is to use a measuring instrument and this is inherently limited by the precision of the measuring instrument. Thus, the fuzziness in the metric is preserved.

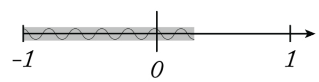

Figure 2: A fuzzy measure encoded in the length of a line segment.

The Ordering Properties

This fuzzification of the evaluation metric leads to a major change in the ordering properties. The usual law of trichotomy is violated and a different method of comparison must be given. It is no longer the case that one is either greater than, lesser than or equal to another; but rather that one is greater than ($>$), lesser than ($<$) or indistinguishable ($\simeq$) from another.

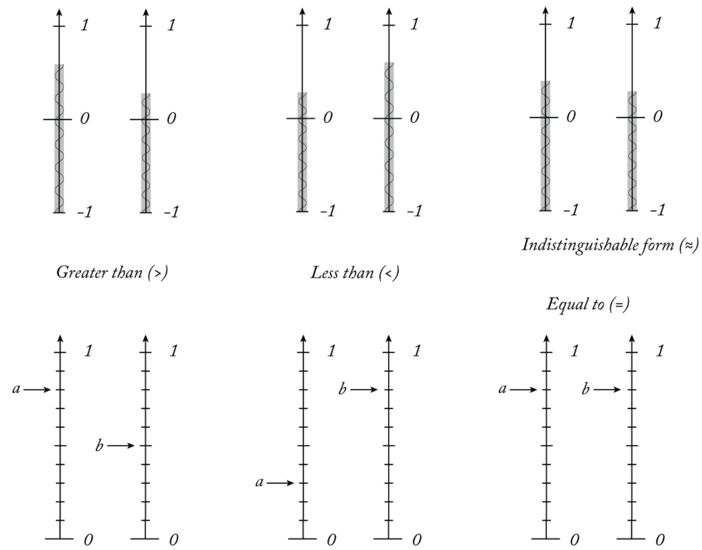

Figure 3: A comparison of ordering among sliders and ordering among discrete measures.

The idea is that it is perfectly clear if something is better or worse than another but not if something is “equal” to another. The closer two values get to being equal, the harder it becomes to distinguish between them. This is directly related to how humans categorise, it becomes harder to distinguish when the said entities entities are more similar. An example could be illustrative. It is easy, almost trivial, to decide if one piece of cake is large than another, but when faced with telling apart cakes which are similar in measure, it can only be said that they’re about the same measure, in other words, indistinguishable.

This is where sliders differ from real number, real numbers do satisfy the usual law of trichotomy because they are mathematical entities that are perfectly precise. Sliders incorporate fuzziness, however, and are thus, by design, imprecise. It could be said that continuity is necessary for fuzziness but not sufficient.

The Algebra

The ordering properties determine how sliders can be compared to one another. However, there would be situations in which the sliders need to interact with each other and combine. These rules of combination are also inspired by the human problem. In addition to being fuzzy, the human method of evaluation is also non-linear and this is the second problem that sliders try to address. The idea is to use a function to map the raw evaluation metric to a slider score using a strictly-increasing, non-linear mapping function. Sigmoid functions are a natural fit in this situation. The sensitivity being the largest in the middle and then tailing off towards the edges seems to be quite characteristic of the way humans evaluate. They have the additional advantage of being bounded.

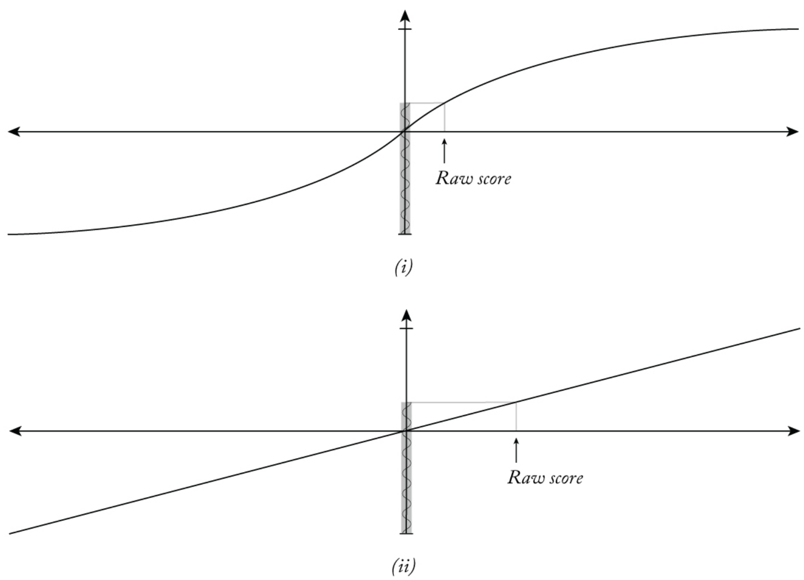

Figure 4: Different mapping functions that can be used for sliders. The raw score is what is measured; it is then mapped to the slider value using a mapping function. (i) A sigmoid mapping function is suitable in most cases. (ii) A linear mapping function is a trivial case.

Slider addition and multiplication are defined in a way such that if the mapping function were the identity, the addition and multiplication would be exactly the same as the usual addition and multiplication. Thus, in order to add or multiply sliders, they are simply mapped back into their raw scores, operated upon and mapped into a new slider.

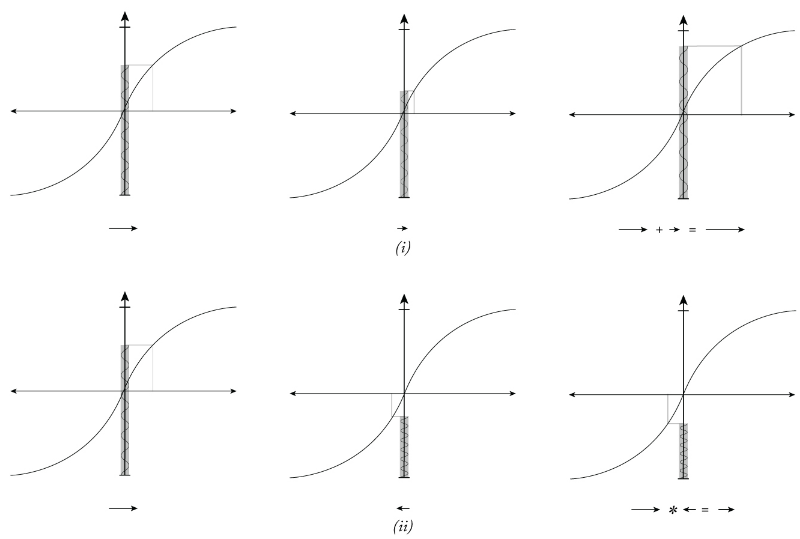

Figure 5: Demonstration of addition and multiplication

If f is the mapping function used on the sliders and a and b are the raw scores, $f(a) = \underline{a}$ and $f(b) = \underline{b}$. Then, $\underline{a} + \underline{b} = f(f^{-1}(\underline{a}) + f^{-1}(\underline{b}))$ and $\underline{a} \times \underline{b} = f(f^{-1}(\underline{a}) \times f^{-1}(\underline{b}))$.

Physical systems that emulate sliders

The Algebra

Qubits are a natural choice for systems that can model sliders. The hidden variable character and probabilistic nature of qubits make them a suitable candidate for encoding sliders. A qubit in a state of $|\psi_a\rangle = \cos(\underline{a})\,|0\rangle + \sin(\underline{a})\,|0\rangle$ naturally models a slider $\underline{a} = \arctan(a)$. Here, a is the raw score and the arctan function naturally serves as a sigmoid mapping function. It is assumed, for now, that sliders are well represented by qubits; this point will be explained in more detail in the next section. Unitaries may be used to perform operations on the sliders to combine them and this would make up the algebra of sliders.

In order to multiply the sliders, one needs to join the qubits (take their tensor product) and then use a suitable unitary to turn it into a slider for the product. Let the original slider qubits be,

$$\left|\psi_{\underline{a}}\right\rangle = \cos(\underline{a})\,|0\rangle + \sin(\underline{a})\,|0\rangle$$
$$\left|\psi_{\underline{b}}\right\rangle = \cos(\underline{b})\,|0\rangle + \sin(\underline{b})\,|0\rangle$$

The result of multiplication should be a slider $\underline{c}$ such that

$$
\begin{aligned}
\underline{c} &= \arctan(\arctan^{-1}(\underline{a}) \times \arctan^{-1}(\underline{b})) \quad = \arctan(\tan(\underline{a}) \times \tan(\underline{b})) \\
\implies \tan(\underline{c}) &= \tan(\underline{a}) \times \tan(\underline{b})
\end{aligned}
$$

$$
\begin{aligned}
\left|\psi_{\underline{a}}\right\rangle \otimes \left|\psi_{\underline{b}}\right\rangle &= (\cos(\underline{a})|0\rangle_a + \sin(\underline{a})|1\rangle_a) \otimes (\cos(\underline{b})|0\rangle_b + \sin(\underline{b})|1\rangle_b) \\
&= \cos(\underline{a})\cos(\underline{b})|0\rangle_a|0\rangle_b + \cos(\underline{a})\sin(\underline{b})|0\rangle_a|1\rangle_b \\
&+ \sin(\underline{a})\cos(\underline{b})|1\rangle_a|0\rangle_b + \sin(\underline{a})\sin(\underline{b})|1\rangle_a|1\rangle_b
\end{aligned}
$$

And so, just multiplying with the unitary

$$
\begin{pmatrix}
1 & & & \\
 & & & 1 \\
 & & 1 & \\
 & 1 & &
\end{pmatrix}
$$

gives the state,

$$
\begin{aligned}
&= \cos(\underline{a})\cos(\underline{b})|0\rangle_a|0\rangle_b + \sin(\underline{a})\sin(\underline{b})|0\rangle_a|1\rangle_b \\
&+ \sin(\underline{a})\cos(\underline{b})|1\rangle_a|0\rangle_b + \cos(\underline{a})\sin(\underline{b})|1\rangle_a|1\rangle_b
\end{aligned}
$$

$$
\begin{aligned}
&= |0\rangle_a \otimes (\cos(\underline{a})\cos(\underline{b})|0\rangle_b + \sin(\underline{a})\sin(\underline{b})|1\rangle_b) \\
&+ |1\rangle_a \otimes (\sin(\underline{a})\cos(\underline{b})\,|0\rangle_b + \cos(\underline{a})\sin(\underline{b})\,|1\rangle_b)
\end{aligned}
$$

Measuring the first qubit and keeping only those that give a $|0\rangle_a$ leads to a qubit with state $\tan(\underline{c}) = \tan(\underline{a}) \times \tan(\underline{b})$ as required. This is because the state is of the form $k \times (\cos(\underline{a})\cos(\underline{b})|0\rangle_b + \sin(\underline{a})\sin(\underline{b})|0\rangle_b)$ and thus the tangent of the "angle" of the slider qubit is

$$
\begin{aligned}
\tan(\underline{c}) &= \frac{k \times \sin(\underline{a}) \times \sin(\underline{b})}{k \times \cos(\underline{a}) \times \cos(\underline{b})} \\
&= \tan(\underline{a}) \times \tan(\underline{b})
\end{aligned}
$$

Additionally, in order to get states that correspond to division, one can keep the states with $|1\rangle_a$ on the first qubit as well leading to states with $\tan(\underline{c}) = \tan(\underline{b})/\tan(\underline{a})$

It is a consequence of quantum mechanics that the sliders only take on the desired values probabilistically. This should not be of consequence however, as it is possible to use the "garbage" values for other purposes.

Addition of sliders can be done in a similar manner, albeit for one complication. We want that,

$$\underline{c} = \arctan(\tan(\underline{a}) + \tan(\underline{b}))$$
$$\implies tan(\underline{c}) = tan(\underline{a}) + tan(\underline{b})$$

$$\begin{aligned}\left|\psi_{\underline{a}}\right\rangle \otimes \left|\psi_{\underline{b}}\right\rangle &= (\cos(\underline{a}) \left|0\right\rangle_a + \sin(\underline{a}) \left|1\right\rangle_a) \otimes (\cos(\underline{b}) \left|0\right\rangle_b + \sin(\underline{b}) \left|1\right\rangle_b) \\ &= \cos(\underline{a}) \cos(\underline{b}) \left|0\right\rangle_a \left|0\right\rangle_b + \cos(\underline{a}) \sin(\underline{b}) \left|0\right\rangle_a \left|1\right\rangle_b \\ &+ \sin(\underline{a}) \cos(\underline{b}) \left|1\right\rangle_a \left|0\right\rangle_b + \sin(\underline{a}) \sin(\underline{b}) \left|1\right\rangle_a \left|1\right\rangle_b\end{aligned}$$

And so, just multiplying with the unitary

$$\begin{pmatrix} 1 & & & \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & & & 1 \\ & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \end{pmatrix}$$

gives the state

$$\left|0\right\rangle_a \otimes \left(\cos(\underline{a}) cos(\underline{b}) \left|0\right\rangle_b + \frac{1}{\sqrt{2}} \times (\cos(\underline{a}) \sin(\underline{b}) + \sin(\underline{a}) \cos(\underline{b})) \left|1\right\rangle_b\right) +$$
$$\left|1\right\rangle_a \otimes \left(\sin(\underline{a}) \sin(\underline{b}) \left|0\right\rangle_b + \frac{1}{\sqrt{2}} \times (\cos(\underline{a}) \sin(\underline{b}) - \sin(\underline{a}) \cos(\underline{b})) \left|1\right\rangle_b\right)$$

If the first qubit is in the state $\left|0\right\rangle_a$ the second qubit is in a state proportional to, $\cos(\underline{a}) \cos(\underline{b}) \left|0\right\rangle_b + \frac{1}{\sqrt{2}} \times (\cos(\underline{a}) \sin(\underline{b}) + \sin(\underline{a}) \cos(\underline{b})) \left|1\right\rangle_b$

This is a slider $\underline{c}$ such that

$$\begin{aligned}\tan(\underline{c}) &= \frac{1}{\sqrt{2}} \times \frac{(\cos(\underline{a}) \sin(\underline{b}) + \sin(\underline{a}) \cos(\underline{b}))}{\cos(\underline{a}) cos(\underline{b})} \\ &= \frac{1}{\sqrt{2}} \times (\tan(\underline{a}) + \tan(\underline{b}))\end{aligned}$$

Multiplying this slider with a slider $\underline{\sqrt{2}} = \cos(\arctan(\sqrt{2})) \left|0\right\rangle + \sin(\arctan(\sqrt{2})) \left|1\right\rangle$ gives a slider that is the sum of the original two sliders.

In case the first qubit is $\left|1\right\rangle_a$ one gets the state,

$$\sin(\underline{a}) \sin(\underline{b}) \left|0\right\rangle_b + \tfrac{1}{\sqrt{2}} \times (\cos(\underline{a}) \sin(\underline{b}) - \sin(\underline{a}) \cos(\underline{b})) \left|1\right\rangle_b$$

This is state with a slider $\tan(\underline{c}) = \frac{1}{\sqrt{2}} \times (\frac{1}{\tan(\underline{a})} - \frac{1}{\tan(\underline{b})})$.

The Outcome

Now that the problem of combination of sliders is solved, how is the use of qubits justified? Its the hidden variable nature of qubits that make them suitable as sliders, the fact that they can only be approximately probed. The additional probabilistic nature of the qubits add another layer of fuzziness.

Comparing the slider values could be done using the unitary,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & & & \frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & & & \frac{-1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \end{pmatrix}$$

One starts with the state,

$$\left|\psi_{\underline{a}}\right\rangle \otimes \left|\psi_{\underline{b}}\right\rangle = (\cos(\underline{a}) \left|0\right\rangle_a + \sin(\underline{a}) \left|1\right\rangle_a) \otimes (\cos(\underline{b}) \left|0\right\rangle_b + \sin(\underline{b}) \left|1\right\rangle_b)$$

And the unitary gives,

$$\left|0\right\rangle_a \otimes (\cos(\underline{a}-\underline{b}) \left|0\right\rangle_b + \sin(\underline{a}-\underline{b}) \left|1\right\rangle_b) + \left|1\right\rangle_a \otimes (\cos(\underline{a}+\underline{b}) \left|0\right\rangle_b + \sin(\underline{a}+\underline{b}) \left|1\right\rangle_b)$$

If the first qubit is $\left|0\right\rangle_a$, the second qubit is in a state of $\cos(\underline{a}-\underline{b}) \left|0\right\rangle_b + \sin(\underline{a}-\underline{b}) \left|1\right\rangle_b$. This state is one suitable to compare sliders. But cannot be natively used as the difference between sliders can vary from $-2*\pi$ all the way to $2*\pi$. It is thus required to first reduce the range of the slider differences.

One gets the state $\cos(\underline{a}+\underline{b}) \left|0\right\rangle_b + \sin(\underline{a}+\underline{b}) \left|1\right\rangle_b$ if the first qubit turns out to be $\left|1\right\rangle_a$.

In order to reduce the range of the slider values it is simply required to divide the difference by 4. In order to do that, one could start with the state $\frac{\cos \underline{c}}{\sqrt{2}} \left|0\right\rangle_p \left|0\right\rangle_r + \frac{\sin \underline{c}}{\sqrt{2}} \left|0\right\rangle_p \left|1\right\rangle_r + \frac{1}{\sqrt{2}} \left|1\right\rangle_p \left|0\right\rangle_r$. This state could be obtained by inputting $\left|+\right\rangle_p$ to a $CSWAP$ gate along with $\left|0\right\rangle_q$ and $\cos \underline{c} \left|0\right\rangle_r + \sin \underline{c} \left|1\right\rangle_r$ on the non-output and output gate respectively.

Using the matrix,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & & \frac{-1}{\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} & & \frac{-1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} & \end{pmatrix}$$

one gets the state $\left(\cos(\frac{\underline{c}}{2}) \left|0\right\rangle + \sin(\frac{\underline{c}}{2}) \left|1\right\rangle\right)^{\otimes 2}$. Applying the same procedure to the two new qubits one gets $\left(\cos(\frac{\underline{c}}{4}) \left|0\right\rangle + \sin(\frac{\underline{c}}{4}) \left|1\right\rangle\right)^{\otimes 4}$. Using the previous unitary one can now get the state $\cos(\frac{\underline{a}-\underline{b}}{4}) \left|0\right\rangle + \sin(\frac{\underline{a}-\underline{b}}{4}) \left|1\right\rangle$ with an angle of $\frac{\underline{a}-\underline{b}}{4}$. The angle now varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$ and thus can be reliably distinguished in the hadamard basis. If the outcome is predominantly $\left|+\right\rangle$, $\underline{a} > \underline{b}$, else if, predominantly $\left|-\right\rangle$, $\underline{a} < \underline{b}$, else, if the outcomes are balanced, $\underline{a} \simeq \underline{b}$. And thus, one obtains a probabilistic, fuzzy measure for the slider.

Conclusion

The concept behind sliders is not something new. This method of dealing with imprecision is already well established in while using units and precision. It is even mundane; high school students regularly use these concepts in their studies. But this has always been viewed as a handicap and never as an advantage. I believe that this is the new perspective that this paper brings about. It uses the idea of imprecision to its own advantage.

In the implementation with qubits it is assumed that the slider values can be encoded into a qubit as many times as required. This would automatically require that the slider cannot be stored as a quantum state. The slider itself must be stored in a "classical" system and must be converted to quantum every time it is required. These are some of the issues that this paper does not get into. And thus there is a lot of scope for this paper to be augmented.

References

[1] Isaac Chuang, Michael Nielsen. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.