Twenty Four

Vishal Johnson

April 2019

Introduction

The inspiration for this paper was a week long seminar at the Indian Statistical Institute, Bengaluru sometime during 2012. Details are hazy; the director of the institute came up to a bunch of us attendees and posed the following question:

Is it possible to come up with ways of combining three "n's" to form 24?

The paper begins with a blatant copying of the ideas that the people around me came up with. Towards the end, there is an attempt at generalisation.

The Easy Cases

The challenge is to use three instances of a number and combine them using common (easily understood) mathematical operators to form Twenty Four. It is best illustrated through example.

Eight Eight is trivial.

$$8 + 8 + 8 = 24$$

Four Four can be done in a few ways. Four is really the key; the reader is implored to pay attention!

$$(4 + 4 - 4)! = 24$$

$$\left(\frac{4 \times 4}{4}\right)! = 24$$

$$\left(4^{\frac{4}{4}}\right)! = 24$$

Three and Five Three and Five are easily obtained from Four.

$$\left(3 + \frac{3}{3}\right)! = 24, \text{ and,}$$

$$\left(5 - \frac{5}{5}\right)! = 24$$

$$3^3 - 3 = 24$$

Nine Nine requires a little bit of extra thought beyond Three.

$$\left(\sqrt{9} + \frac{9}{9}\right)! = 24$$

Six

$$\left(6 - log_{\sqrt{6}}6\right)! = 24$$

Two and Seven and... Once the idea of logarithms sets in, it becomes quite easy to generalise.

$$log_{\sqrt{2}}\,(2 \times 2)! = 24$$

$$22 + 2$$

$$log_{\sqrt{7}}\,(7 \times 7)! = 24$$

But this would work for almost any positive real number! In fact this idea can be extended to negative numbers as well!

$$log_{\sqrt{|r|}}\,(|r| \times |r|)! = 24, r \in \mathbb{R}, |r| \neq 0, 1$$

But we could keep going further! This would even work for most complex numbers.

$$log_{\sqrt{\|z\|}}\,(\|z\| \times \|z\|)! = 24, z \in \mathbb{C}, \|z\| \neq 0, 1$$

Of course, the difficulty arises with Zero and One, and a discussion of this is what would occupy the next section.

The Harder Cases

A solution for One seems so close at hand but yet so far. The reason, of course, is that One is special. It is very hard to start with One and end up with a non-special number. It seems even harder with Zero. The director's solution was quite brilliant and innovative. However, in my experience, many find this be a cheat answer. Thus, an attempt will be made to come up with a slightly less elegant, slightly less offensive solution.

Elegant Solution

$$\left(\sum (1 + 1) + 1\right)! = 24$$

$$\left(\sum (0! + 0!) + 0!\right)! = 24$$

The reader shall be a judge of whether or not this constitutes cheating, the article proceeds to demonstrate the other trick.

Ugly Solution

$$\left(\frac{1}{\int_{\arccos 1}^{\arctan 1} cos(x)sin(x)dx}\right)! = 24$$

$$\left(\frac{0!}{\int_{\arcsin 0}^{\arctan 0!} cos(x)sin(x)dx}\right)! = 24$$

Here, x is a dummy variable.

Entire Solution

$$log_{\sqrt{\|z\|}}(\|z\| \times \|z\|)! = 24, z \in \mathbb{C}, \|z\| \neq 0, 1$$

$$\left(\frac{\|z\|}{\int_{\arccos \|z\|}^{\arctan \|z\|} cos(x)sin(x)dx}\right)! = 24, \|z\| = 1$$

$$\left(\frac{\|z\|!}{\int_{\arcsin \|z\|}^{\arctan \|z\|!} cos(x)sin(x)dx}\right)! = 24, \|z\| = 0$$

In fact this idea can easily be generalised to any normed vector space. This seems to be quite general!

Uglier Generalisation

Consider any number of copies of any member of a normed vector space; even works for zero copies. Let the member of the vector space be z and the number of copies be n.

$$\left(\frac{(y-y)!}{\int_{\arcsin (y-y)[\times\|z\|]^n}^{\arctan (y-y)!} cos(x)sin(x)dx}\right)! = 24$$

It is emphasised that x and y are just dummy variables. $[\times\|z\|]^n$ just means: "insert n copies of $\times\|z\|$ here".

Again, the reader shall be the judge of whether this is cheating.

Conclusion

It may seems like a philosophically void exercise to go through all this trouble to show something specific to Twenty Four. All one has to do, is reverse the digits to realise its cosmic significance!

Acknowledgement

Thanks to the director of IISc, Bengaluru and all the folks around when this transpired. Thanks are also due to the many people I've discussed this over the years and whose opinions helped shape the form of the solution.